Valuing Government Obligations When Markets are Incomplete*

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Abstract

This paper posits and simulates a ten-period overlapping generations model with aggregate shocks to price safe and risky government obligations using consumption-asset pricing. Agents can't trade with future generations to hedge the model's productivity and depreciation shocks, and can only invest in one-period bonds and risky capital. We find that the pricing of short- and long-dated riskless obligations is anchored to the prevailing one-period risk-free return. The prices of obligations whose values are proportional to the prevailing wage are essentially identical to those of safe obligations, notwithstanding large macro shocks. On the contrary, government obligations provided in the form of options entail significant risk adjustment. We also show that the value of obligations to unborn generations depends on the nature of the compensating variation. Our model suggests the potential of CGE OLG models to price government obligations and non-marketed private securities in the presence of incomplete markets and macro shocks.

Keywords: Government Discount Rates; Aggregate Risk; Incomplete Markets.

JEL Classification: E62; H55; H31; D58; G12; C68.

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1 Introduction

Properly valuing government commitments to pay benefits to or extract taxes from households is a longstanding issue in public finance. It figures prominently in cost-benefit analysis, in valuing liabilities of government pension systems, and in assessing government inter- and intra-generational redistribution. Were markets complete, one could simply check the prevailing price of a given benefit or tax in a given contingent state. But markets are far from complete for many reasons including the inability of the living to trade with the unborn. Economists have attempted to overcome the missing-market problem by applying arbitrage pricing theory (APT) and treating government obligations as derivatives on marketed assets – assets that arguably span the government's promised payments, be they positive, negative, safe or risky. Unfortunately, the assets/factors needed to span government obligations may not exist. Alternatively, the specification of the spanning relationship may materially alter the arbitrage pricing.[1]

An alternative approach, and that taken here, is structural general-equilibrium modeling, which uses consumption-asset pricing to value non-marketed government securities. Such pricing is based on marginal compensating differentials – the extra current consumption needed to compensate agents for forgoing future safe or risky net government payments. This approach, based on remaining expected lifetime utility, fully captures the non-linear general equilibrium (GE) response of the economy to shocks. In so doing, it eliminates the guesswork in specifying the nature of government payment risk, albeit at the price of potentially mis-specifying the structural model.

Fortunately, recent computational advances permit simulating life-cycle models with many periods, significant macro shocks, and alternative government policies. Krueger and

[1]For example, Lucas and Zeldes (2006) and Geanakoplos and Zeldes (2010, 2011) postulate a different and arguably more realistic reduced-form structure connecting lagged returns on a Social Security wage-growth security, which pays off based on the realized growth rate of the wage, and stock returns than do Blocker, Kotlikoff and Ross (2008). These differences in specifications produce significant differences in the valuation of Social Security's net liabilities.

Kubler (2006) represents a major milestone in this direction. They consider a 9-period overlapping generations model with aggregate risk and one-period bond market. Aggregate risk is measured as trend stationary and enters the model via shocks to total factor productivity and stochastic rate of depreciation. Production is Cobb-Douglas in capital and exogenously supplied labor. To solve the model, the authors overcome the curse of dimensionality by applying Smolyak's (1963) algorithm to efficiently choose a small (sparse) set of grid values over the state space.[2] This allows them to *quantify*, for the first time, the extent to which the introduction of an unfunded Social Security system provides a Pareto improving policy reform – a question of ongoing policy debate.

Our model is a 10-period variant of Hasanhodzic and Kotlikoff's (2013, 2017) 80-period life-cycle model with aggregate risk and a one-period safe bond. As in Krueger and Kubler (2006), the model is trend stationary, with shocks to both total factor productivity and capital's rate of depreciation. Also, as in that paper, production is Cobb-Douglas and labor is exogenously supplied. Preferences over the single commodity are time separable and isoelastic. Prior to their births, future generations are assumed to have expected utility arising from consumption after they are born. Specifically, we assume an agent age minus τ discounts utility from consumption (when alive) by an extra τ years – the time it takes for her to be born.

The Hasanhodzic-Kotlikoff solution algorithm builds on Marcet's (1988) method, which was further operationalized by Marcet and Marshall (1994) and Judd, Maliar, and Maliar (2009, 2011). The method overcomes the curse of dimensionality by solving for decision functions only in states that fall within the economy's ergodic set; i.e., in states that the economy will frequent, not those that it will essentially never visit.

We find that the pricing of both long- and short-dated riskless government payment promises are closely anchored to the prevailing one-period risk-free return. Depending on

[2] Krueger and Kubler (2004) and Malin, Krueger, and Kubler (2011) detail this method. Brumm and Scheidegger (2017) propose an adaptive sparse grid approach to efficiently solve high-dimensional dynamic models, although their applications do not include an OLG model.

their payoff durations, these assets can be priced by discounting the future net payment at rates either at or fairly close to the prevailing risk-free rate. Since the current return on risk-free bonds can differ dramatically from the average risk-free return, our findings suggest the importance of pricing government promises based on prevailing, not average historic market returns.[3] More surprising, short- and long-dated risky government payments, whose amounts are proportional to the prevailing risky wage, are priced quite similarly to safe pension promises, i.e., there is little risk adjustment. This is true notwithstanding our model's large macro shocks. On the other hand, we show that government obligations in the form of options are priced with a substantial risk premium. The message in these examples is that risk-adjusted pricing is highly specific to the risk. We also find that pricing government promises to the unborn, whether safe or risky, depends on the manner in which compensation is provided. Another result involves the importance of the one-period bond market to valuing obligations. The presence of the bond market matters, although less than one might expect. Finally, we can use data generated by our model to explore the ability of arbitrage pricing to get the prices right. We show that, with the right spanning assumptions, arbitrage pricing theory (APT) can do remarkably well.

Our model is intentionally bare-bones to make qualitative, not precise quantitative points. Its GE consumption-asset pricing is very different from the APT-based pricing frameworks of Lucas and Zeldes (2006), Santos and Veronesi (2006), Goetzmann (2008), Blocker, Kotlikoff, and Ross (2008), Khorasanee (2009), and Geanakoplos and Zeldes (2010, 2011). Although we demonstrate that with the right reduced form, a GE structural or a reduced-form APT approach can correctly price government I.O.U.s, we also show that specifying the wrong APT reduced form can produce mis-pricing. Hence, the use of APT introduces an element of risk not present in structural modeling. Of course, specifying the wrong structural model also raises the risk of mis-pricing. Our goal here is not to adjudicate the two approaches. It's

[3]This point is particularly important for Social Security's actuaries who, in the annual Social Security Trustees Report, discount the system's liabilities using an historic average real return.

simply to suggest the potential for using large-scale CGE OLG models to price government obligations and, indeed, private, non-marketed securities.

The next section reviews a small portion of the voluminous relevant literature. Subsequent sections present our model, describe its solution, discuss its calibration, examine the precision and nature of our findings, and draw conclusions.[4]

2 Related Studies

There is a large literature concerning the proper means to value government net payment promises, be they pension obligations, tax assessments, or returns from government investments.[5] Lucas (2014) reviews key contributions and discusses the policy stakes involved. As she points out, the sixties and early seventies witnessed a major debate over the proper government discount rate. Hirschleifer (1964, 1966) argued for risk-adjusted discounting in valuing government investments. Arrow and Lind (1970) argued for risk-free discounting based on the government's assumed superior ability to diversify idiosyncratic shocks and the proposition that government investments are uncorrelated with macro risk.

Lucas (2014) sides with Hirschleifer, referencing the failure to detect idiosyncratic risk in the pricing of securities. This as well as the myriad opportunities households have to diversify investments implies that government promises to deliver future dollars should be discounted at a higher rate the greater the risk of the payoff. Still, the "old" literature as well as Lucas' discussion makes clear that proper pricing of government promises, including investment returns, depends on the financial-market-completing aspects of those promises together with the other economic factors.[6]

[4] The model, except for the number of time periods, is identical to that in Hasanhodzic and Kotlikoff (2017). Hence, we borrow freely from that paper in presenting the model and its solution method.

[5] This literature review borrows freely and often verbatim from the literature review in a related paper, namely Hasanhodzic and Kotlikoff (2013, revised 2018).

[6] As Lucas (2012, 2014) stresses, getting the prices right matters. If, for example, governments borrow to invest in risky assets, but treat their future returns as risk-free, they will, as Lucas puts it, falsely claim to have "a free money machine" and over invest. Disregarding risk can also lead to the underfunding and over provision of government pensions as well as an understatement of the true costs of government credit

Clearly, having a fully specified CGE model permits precise consumption-based pricing of government promises. Such modeling can, we believe, extend far beyond the highly stylized framework presented here. This said, precisely how much detail models like ours can accommodate remains to be seen. The alternative to structurally pricing government promises is reduced-form, empirical pricing based on Ross' (1976a, 1976b) Arbitrage Pricing Theory (APT) and its associated risk-neutral, derivative-pricing and process-free pricing theories.[7] Lucas and Zeldes (2006) is an early paper that applies modern asset-pricing theory and APT techniques to value pension promises in a realistic setting. Their focus is on private-sector defined benefit pensions. But their approach extends automatically to government-provided pensions.

Blocker, Kotlikoff, and Ross (2008) also use risk-neutral derivative pricing to value pensions. Their work differs from Lucas and Zeldes (2006) in two ways. First, they focus on Social Security's benefit and tax promises. Second, they relate the growth rate of wage rates to current only or one-period-only lagged asset returns. They find very similar pricing of wage growth rate securities regardless of the choice of contemporaneous or lagged regressors. But their lagged regressors produce a much higher R-Squared.[8]

In contrast, Lucas and Zeldes (2006) posit a diffusion process for wages and stock values, which produces a small contemporaneous but significant long-term correlation between earnings growth and stock returns. They justify their assumed process based on Goetzmann's (2005) finding of a low annual correlation between aggregate wage growth and stock returns. Geanakoplos and Zeldes (2010, 2011) also value Social Security promises using a diffusion process, pointing to Benzoni et al. (2007) as providing additional support for the assumption of low short-run, but high long-run correlations between wage growth rates and stock

programs.

[7]See Cox and Ross (1976), Cox, Ingersol, and Ross (1977), Ross (1978), and Cox, Ross, and Rubinstein (1979). We say "reduced form" because the operationalization of this pricing method requires positing and estimating how government promises co-vary with either marketed assets or a subset of economic factors meant to capture undiversifiable economic risk.

[8]0.454 compared to 0.105

returns.[9]

The different approaches generate quite different assessments of Social Security's unfunded liabilities. Blocker et al. (2008) find a significant understatement of these liabilities. Social Security's mistake, they argue, is not its failure to adjust for risk, but its failure to adjust for safety. Social Security's actuaries discount benefits, once they've been received and become sure liabilities, at a rate far above the market rate on long-term TIPS (Treasury Inflation Protected Securities). Although Geanakoplos and Zeldes (2010, 2011) report that Social Security's liabilities are significantly overstated, they adopt Social Security's overly high safe discount rate. Hence, their measure of Social Security's liabilities, while it may be closer to the mark than Blocker et al. (2008), appears biased downward.

The above said, our model provides a special opportunity to test APT in a controlled manner. As we show, with the right APT specification, APT pricing does an excellent job in approximating consumption-asset pricing based on our structural model. We also show, however, that mis-specifying the reduced form or forming APT valuations based on average, rather than contemporaneous returns, can be problematic.

3 The Model

Our model features $G = 10$ overlapping generations with total factor productivity and capital depreciation shocks. Each agent works full time through retirement age $R = 7$, dies at age G, and maximizes expected lifetime utility. Cohort members are identical. Each cohort supplies 1 unit of labor each period when working. Hence, total labor supply equals the

[9] Geanakoplos and Zeldes (2011) provide an interesting Lucas tree-type model involving the early revelation of news about future productivity shocks. This information acquisition produces zero short-run but high long-run correlation between wage growth and stock returns. Their model is, however, very different from ours. Our model incorporates capital accumulation and decumulation. And its current return to capital as well as its current wage growth is fully determined by current economic conditions.

retirement age R. Utility is time-separable and isoelastic with risk aversion coefficient γ.

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}. \tag{1}$$

Production is Cobb-Douglas with output, Y_t, given by

$$Y_t = z_t K_t^{\psi} L_t^{1-\psi}, \tag{2}$$

where z is total factor productivity, ψ is capital's share of output, K_t is capital, and L_t is labor demand, which equals labor supply, R. Equilibrium factor prices satisfy

$$w_t = z_t(1-\psi)\left(\frac{K_t}{R}\right)^{\psi}, \tag{3}$$

$$r_t = z_t\psi\left(\frac{K_t}{R}\right)^{\psi-1} - \delta_t, \tag{4}$$

where depreciation $\delta_t \sim \mathcal{N}(\mu_\delta, \sigma_\delta^2)$ as in Ambler and Paquet (1994). Total factor productivity, z, obeys

$$\ln(z_{t+1}) = \rho \ln(z_t) + \epsilon_{t+1}, \tag{5}$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

3.1 Financial Markets

Households save and invest in either risky capital or one-period safe bonds. Investing 1 unit of consumption in bonds at time t yields $1 + \bar{r}_t$ units in period $t+1$. The safe rate of return,

$\bar{r}_t$, is indexed by t since it is known at time t although it is received at time $t+1$. Bonds are in zero net supply. Hence, households that are short (long) bonds, are borrowing (lending) to one another. The asset holdings of agents age g at time t is denoted by $\theta_{g,t}$ and their share of assets invested in bonds is denoted by $\alpha_{g,t}$. Households enter period t with $\theta_{g-1,t-1}$ in assets, which corresponds to the total assets they demanded the prior period. Since investment decisions are made at the end of the period, the aggregate supply of capital in period t, K_t, is the sum of assets brought by the households into period t, i.e.

$$K_t = \sum_{g=1}^{G} (1 - \alpha_{g,t})\theta_{g-1,t-1}. \tag{6}$$

Bonds are in zero net supply[10], hence for all t,

$$\sum_{g=1}^{G} \alpha_{g,t}\theta_{g,t} = 0. \tag{7}$$

3.2 Social Security

Our model includes a pay-as-you-go Social Security system.[11] Each retiree receives a benefit each period equal to 0.35 times that period's wage. This equals 0.15 times 7 divided by 3 reflecting our assumed 15 percent payroll tax rate and our model's 7 workers per 3 retirees. Hence, letting $H_{g,t}$ denote the tax levied on the age-g household at time t and $B_{g,t}$ denote the benefit paid to the age-g household at time t, we have

[10] As shown in Green and Kotlikoff (2008), fiscal policy can be labeled in an infinite number of ways to produce whatever time path of explicit and implicit debts the government wishes to report. Such relabeling makes no difference to this or any other neoclassical model, i.e., all relabeled models are isomorphisms. Hence, our model can be viewed as including government debt or not depending on the reader's preferences. With government debt included in the policy's labeling, the left-hand-side of (7) would be larger by the amount of debt. But the right-hand-side would also be larger by exactly the same amount, leaving the capital stock unchanged.

[11] Although risk-free and risky returns are lower without Social Security, including Social Security makes no difference to our findings.

$$B_{g,t} = \begin{cases} 0.35 \times w_t & \text{for } g \in \{8, 9, 10\} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and

$$H_{g,t} = \begin{cases} 0.15 \times w_t & \text{for } g \in \{1, 2, \ldots, 7\} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

3.3 Household Problem

At time t the economy's state is s_t, where s_t references the vector $(z_t,\ \delta_t,\ x_{1,t},\ \ldots,\ x_{G-1,t})$ and $x_{g,t}$ references cash on hand of an agent age g at time t.[12]

$$V_g(s_t) = \max_{c_{g,t},\alpha_{g,t}} \left\{ u(c_{g,t}) + \beta \mathrm{E} \left[V_{g+1}(s_{t+1}) \right] \right\} \quad \text{for} \quad g < G, \ \text{and} \tag{10}$$

$$V_G(s_t) = u(c_{G,t}), \tag{11}$$

[12] Note that $\sum_g x_{g,t} = K_t(1 + r_t) + w_t R$. This equation plus equations 3 and 4 suffice to solve for K_t, w_t, and r_t. Hence, K_t isn't needed to define the state.

subject to

$$c_{1,t} = \ell_1 w_t - \theta_{1,t} - H_{1,t} + B_{1,t}, \tag{12}$$

$$c_{g,t} = \ell_g w_t + \left[\alpha_{g-1,t-1}(1+\bar{r}_{t-1}) + (1-\alpha_{g-1,t-1})(1+r_t)\right]\theta_{g-1,t-1} - \theta_{g,t} - H_{g,t} + B_{g,t}, \tag{13}$$

for $1 < g < G$, and

$$c_{G,t} = \ell_G w_t + \left[\alpha_{G-1,t-1}(1+\bar{r}_{t-1}) + (1-\alpha_{G-1,t-1})(1+r_t)\right]\theta_{G-1,t-1} - H_{G,t} + B_{G,t}, \tag{14}$$

where $c_{g,t}$ is the consumption of a g-year old at time t, and (12)–(14) are budget constraints for age group 1, those between 1 and G, and for age group G. With the above definitions of $c_{g,t}$, cash on hand is simply defined as $x_{g,t} = c_{g,t} + \theta_{g,t}$.

3.4 Equilibrium

For any state s, the recursive competitive equilibrium is defined as follows.

Definition. The recursive competitive equilibrium is governed by the consumption functions, $c_g(s)$, the share of saving of the young invested in bonds, $\alpha_g(s)$, factor demands of the representative firm, $K(s)$ and $L(s)$, Social Security policy as well as the pricing functions $r(s)$, $w(s)$, and $\bar{r}(s)$ such that:

1. Given the pricing functions, the value functions (10) and (11) solve the recursive problem of the households subject to the budget constraints (12)–(14), and θ_g, α_g, and c_g are the associated policy functions for all g and all dates and states.

2. Wages and rates of return on capital satisfy (3) and (4).

3. The government budget constraint (8) is satisfied.

4. All markets clear.

5. For all age groups $g = 1, \ldots, G-1$, optimal intertemporal consumption and investment choice satisfies

$$1 = \beta E_t \big[\big(1 + r(s_{t+1})\big) \frac{u'(c_{g+1}(s_{t+1}))}{u'(c_{g,t}(s_t))} \big] \tag{15}$$

and

$$0 = E_t \left[u'(c_{g+1}(s_{t+1})) \big(\bar{r}(s_t) - r(s_{t+1})\big) \right], \tag{16}$$

where E_t is the expectation operator.

4 Calibration

The parameters, apart from our assumed 15 percent payroll tax, τ, are calibrated as follows.

4.1 Endowments and Preferences

As indicated, agents work for $R = 7$ periods and live for $G = 10$. This corresponds to real life ages 20 to 80, so each period in our model represents 6 years. We set the quarterly subjective discount factor, β, to 0.99. This implies a six-year value of 0.786 for β. Risk aversion γ equals 3.

4.2 Technology

We calibrate the TFP process, z, based on Hansen (1985) and Prescott (1986).[13] Hansen estimates a quarterly value for the autocorrelation coefficient, ρ, of 0.95 and a standard deviation, σ, of the innovation ϵ ranging from 0.007 to 0.01. Prescott's (1986) estimates are 0.9 for ρ and 0.00763 for σ.

[13]This TFP formuation is standard. See, e.g., Cooley and Prescott (1995), Ríos-Rull and Santaeulalia-Llopis (2010), Gomme, Rogerson, Rupert, and Wright (2005), and Judd, Maliar, and Maliar (2011).

Our assumed quarterly values for ρ and σ are 0.95 and 0.01, respectively. On a six-year basis they are 0.292 and 0.031, respectively, generating a mean TFP value of 1.000 with a standard deviation of 0.032. We set the quarterly value of the standard deviation, ψ, of the depreciation shock, δ, to 0.0144 (implying a six-year value of 0.346).[14] This is higher than the 0.0052 quarterly estimate of Ambler and Paquet (1994). This size depreciation shock was chosen to generate variability in risky returns that resembles that in the return on U.S. aggregate wealth.

With this calibration of the shocks, the annualized rate of return on capital displays a standard deviation of 4.211 percent, around a mean of 7.023 percent. This accords fairly well with the return on aggregate U.S. wealth in the data, which is characterize by an annual standard deviation of 4.886 percent and a mean of 6.512 percent.[15] Moreover, with this calibration the wage displays a standard deviation of 0.050 around a mean of 0.500, for a coefficient of variation of 10 percent.

Finally, the capital share of output, ψ, equals 0.33.

5 Solution Method and Its Precision

Our algorithm contains outer and inner loops. The outer loop solves for consumption functions of each generation. The inner loop uses a combination of techniques from the numerical analysis literature – Broyden, Gauss-Seidel, and Newton's method – to compute the agents' bond holdings and the risk-free rate that clears the bond market.

[14]We interpret Y (equation 2) as the net production function, and hence set the mean value of depreciation to zero.

[15]To measure the empirical equivalent to the model's return on capital we use the national income accounting identity that $W_{t+1} = W_t + r_t W_t + E_t - C_t - G_t$, where W_t stands for national wealth at time t , E_t stands for labor income at time t, C_t standards for household consumption at time t, and G_t stands for government consumption at time t. We solve this identity for annual values of r_t by plugging in values of W_t, reported in the Federal Reserve's Financial Accounts data, and E_t, C_t, and G_t, reported by the Bureau of Economic Analysis in the National Income Accounts. Our data for this calculation cover 1947-2015. All data were converted into real dollars using the PCE index and measured at producer prices. The share of labor earnings in proprietorship and partnership income was assumed to equal the overall share of labor income to national income on a year-by-year basis.

Recall that the state vector consists of cash-on-hand variables, $x_{g,t}$, of generations 1 through $G-1$ and exogenous shocks. Given the information at time t, agents decide how much of their cash on hand to consume, $c_{g,t}$. They also choose the proportion $\alpha_{g,t}$ of their savings to allocate to bonds at the prevailing risk-free rate $\bar{r}_t$. The outer loop starts by making an initial guess of stationary generation-specific consumption functions, c_g, as linear polynomials in the state vector and the prevailing depreciation shock.[16] Next, we take a draw of the path of shocks for $T = 600$ periods. We then run the model forward for T periods using the economy's initial conditions (which corresponds to the non-stochastic steady state of the no-policy model), guessed consumption functions and the drawn shocks. I.e., we compute cash-on-hand variables at time $t+1$ using the information we have at time t and the exogenous shocks at time $t+1$.[17]

At each time t, we compute the agents' choice of bond shares and the risk-free rate that clears the bond market. To solve for $\bar{r}_t$, we use Broyden's method based on the bond-market clearing condition (equation 7). This condition requires that the sum of bond holdings at time t equals zero. The bond holdings at time t of each agent age g is $\alpha_{g,t}\theta_{g,t}$. The choice of the $\alpha_{g,t}$'s make them functions of $\bar{r}_t$. Hence, for given values of the $\theta_{g,t}$'s, the bond-market clearing condition is a function of $\bar{r}_t$ and can be used, via Broyden's method, to find the $\bar{r}_t$ that sustains market clearing.

For any given $\bar{r}_t$, the choice of $\alpha_{g,t}$'s is determined by Gauss-Seidel iterations to solve the system of simultaneous $G-1$ generation-specific Euler equations governing the choices of the $G-1$ α's for the new values of those α's. Specifically, for given guesses of each agent's value of α, other than that of agent i, we apply Newton's method to agent i's Euler equation to determine the new guessed value of α for agent i.[18]

Simulating the model forward produces the data needed to update our guessed consump-

[16] Although we do not include δ as part of the theoretical state space, using it as a regressor for approximating the consumption functions proved useful.

[17] The α's and the $\bar{r}$, which are determined at time t, are used to compute each age cohort's cash-on-hand in period $t+1$.

[18] Taking other unknowns as given is Gauss-Seidel.

tion functions. Specifically, for each age group g and each period t, we evaluate the Euler condition to determine what that age group's consumption should be in that period. This calculation is based on the derived period-t state variables and the current guessed consumption functions of all age groups. These functions determine each age-g agent's marginal utility of consumption at $t+1$.

Following Judd, Maliar, and Maliar (2009, 2011), we then regress these time series of age-specific consumption levels on the state variables plus the depreciation shock using least squares with Tikhonov regularization. We use the new regression estimates to update, with dampening, the polynomial coefficients of each guessed consumption function. We iterate the updating of these functions based, always, on the same draw of the path of shocks until all consumption functions converge. We evaluate the accuracy of our solutions using two methods proposed in the literature – out-of-sample deviations from the exact satisfaction of the Euler equations and the statistic proposed by Den Haan and Marcet (1989, 1994). We also consider whether each age group accurately prices safe assets dated one period in the future.

5.1 Out-of-Sample Deviations from the Perfect Satisfaction of Euler Equations

A satisfactory solution requires that generation-specific Euler equations (15) hold out of sample. Hence, to test the accuracy of our solution, we draw a fresh sequence of 2000 sets of shocks for each simulated model. We then run the model forward for 2000 years, imposing the drawn shocks, using the original consumption functions, c_g, and clearing the bond market by rerunning the model's inner loop each year as we move through time. To calculate out-of-sample, unit-free deviations from full satisfaction of the Euler equations, we form

$$\epsilon(s_{g,t}) = \beta E_t\big[\big(1 + r(s_{t+1})\big)\frac{u'(c_{g+1}(s_{t+1}))}{u'(c_{g,t}(s_t))}\big] - 1 \tag{17}$$

for each period in the newly simulated time path and for each generation $g \in \{1, \dots, G-1\}$. Finally, we compute the average, across time, of the absolute value of the deviations from these Euler equations for each generation.

The top panel of table 1 reports summary statistics, across generations, of their average absolute deviations from Euler equations for our model with and without the bond market.[19] As indicated, in all cases these deviations are at most 0.005.

The portfolio choice equations (16) and the bond market-clearing condition (7) hold very precisely by construction, since the α's and $\bar{r}$ that satisfying them are calculated in the inner loop with a high degree of precision. In particular, the average absolute deviations from these equations, which theoretically should equal zero, are at most 3×10^{-7} and 9.8×10^{-5}, respectively, and in most cases are smaller by an order of magnitude.

5.2 The Den Haan-Marcet Statistic

An alternative precision test is provided by Den Haan and Marcet (1989, 1994). Taylor and Uhlig (1990) use this test to compare alternative solution methods for nonlinear stochastic growth models. We follow Taylor and Uhlig's particular implementation method.

As above, we start with a fresh draw of shocks over T periods and simulate the model forward based on these shocks, using the original consumption functions and clearing the bond market each period based on the inner loop technique (discussed above). We set T to 1200, twice the length of the original simulation. Then, for each generation-specific Euler

[19] Note, these deviations are not Euler errors, which capture differences in period t's marginal utility and period $(t+1)$'s realized marginal utility (properly weighted by β and $r(s', z')$). Rather, they reference mistakes in satisfying the Euler equation, i.e., the discrepancy in period t between the marginal utility and its properly weighted time-t expectation.

Solution Precision

	Min	Mean	Max
Mean Absolute Euler Equation Deviations			
Bond Market	0.003	0.004	0.005
No Bond Market	0.002	0.003	0.005
Den Haan-Marcet Statistic			
Bond Market	4.521	9.613	12.410
No Bond Market	5.151	10.360	16.236

Table 1: The minimum, mean, and maximum values, calculated across generations, of the average, calculated across time, of the absolute generation-specific, out-of-sample deviations from perfect satisfaction of the Euler equations are reported in the top panel. The bottom panel reports the minimum, mean, and maximum values across generations of the Den Haan-Marcet statistic. The model's precision statistics are computed in both the presence and the absence of the bond market.

equation (15), we compute η_g, where g references the generation's age at time t:

$$\eta_g(t) = \beta\big(1 + r_{t+1}\big)\frac{u'(c_{g+1,t+1})}{u'(c_{g,t})}. \tag{18}$$

We next regress, separately for each generation, their 1200 η_g values on a matrix x_g consisting of a constant, five lags of c_g, and five lags of z. The regression coefficients, $\hat{a}_g$,

$$\hat{a}_g = (\Sigma x_g(t)' x_g(t))^{-1}(\Sigma x_g(t)'\eta_g(t)), \tag{19}$$

are then used to construct the Den Haan-Marcet statistic m_g as follows:

$$m_g = \hat{a}_g'\big(\Sigma x_g(t)' x_g(t)\big)\big(\Sigma x_g(t)' x_g(t)\eta_g(t)^2\big)^{-1}\big(\Sigma x_g(t)' x_g(t)\big)\hat{a}_g. \tag{20}$$

If the generation-specific Euler equations (15) are satisfied, then $E_{t-1}[\eta_g(t)] = 0$ must hold. This implies that the coefficient vector, and, therefore, m_g are zero, which is the null hypothesis. Note that our solution method does not enforce this property, so as Den Haan and Marcet (1994) point out, theirs is a challenging test.

Under the null, m_g is distributed as $\chi^2(11)$ asymptotically. Based on a two-sided test at the 2.5 percent significance level, we would reject the null if m_g lies outside the interval (3.82, 21.92). In the bottom panel of Table 1 we compute the minimum, mean, and maximum across generations of generation-specific statistics m_g for our model in both the presence and the absence of the bond market. In all cases, the mean across generations of the statistic is well within the acceptance interval.

5.3 Discount Rates for Pricing a One-Period Goverment Payment Promise

Yet a third way to test for our model's accuracy is to consider our derived rates for discounting one-period-ahead safe government payments. Combining equations 15 and 16 yields equation 21:

$$1/(1+\bar{r}(s_t)) = \beta E_t\Big[\frac{u'(c_{g+1,t+1}(s_{t+1}))}{u'(c_{g,t}(s_t))}\Big]. \tag{21}$$

This equation states that all generations at time t price a sure payment made a period from the present at $1/(1+\bar{r}(s_t))$. In the asset-pricing tables presented below, the first column presents the derived one-period ahead discount rates for generations of different ages. Since our model's solution doesn't directly incorporate equation 21, differences in the values in the first columns imply differences in $1/(1+\bar{r}(s_t))$ and represent a third measure of the model's goodness of fit. As will be apparent, there are differences in discount rates in the first columns. For example, in our first pricing table, Table 4, the prevailing value for $\bar{r}(s_t, z_t)$ is 0.050 on an annualized basis. But one value in the column equals 0.048. This represents a four percent difference in annualized discount rates, but only a 1.15 percent difference in the price of the security.[20]

6 Valuing Government Promises

We first review consumption-asset pricing of government payments to the living and then extend this method to pricing obligations to the unborn.

6.1 Valuing Government Payment Promises to the Living

Equation 22 considers the impact on the remaining lifetime utility of an agent age g in year t of paying $m_{g,t}$ units of consumption to the government in period t and receiving $\tilde{\epsilon}_{t+\tau} \times \bar{P}_{t+\tau}$ from the government in period $t+\tau$, where the later term is the average value of the payment and the former term, which has a mean of 1, is its random component. Equation 23 calculates how much $m_{g,t}$ needs to increase to compensate for the receipt of the risky $t+\tau$ payment. This is marginal consumption-asset pricing. It tells us how much additional current consumption is needed to compensate the agent (and, thereby, maintain her expected remaining lifetime utility) for foregoing the future government payment.[21] Note that if $\tilde{\epsilon}_{t+\tau} = 1$ for all τ, the government's payment is a safe asset. Equation 24 relates the implied per period discount rate, μ, to the price of the asset and equation 26 annualizes it.

$$\begin{aligned} EU_{g,t} &= u(c_{g,t} - m_{g,t}) + \beta E_t[u(c_{g+1,t+1})] + \dots \\ &\quad + \beta^{\tau} E_t[u\big(c_{g+\tau,t+\tau} + \tilde{\epsilon}_{t+\tau} \times \bar{P}_{t+\tau}\big)] + \dots + \beta^{10-g} E_t[u(c_{10,t+10-g})], \end{aligned} \tag{22}$$

[20] This figure is calculated by dividing $(1/1.048)^6$ by $(1/1.05)^6$. Indeed, across all the tables presented below, the largest discrepancy across generations in the pricing of a one-period safe security is 2.18 percent.

[21] Equivalently, $m_{g,t}$ is the reduction in current consumption needed to offset the provision of the risky payment at $t+\tau$. This derivative, evaluated at $\bar{P}_{t+\tau}$, equals zero.

$$\frac{dm_{g,t}}{d\bar{P}_{t+\tau}} = \beta^{\tau} \frac{E_t[u'(c_{g+\tau,t+\tau}) \times \tilde{\epsilon}_{t+\tau}]}{u'(c_{g,t})}, \tag{23}$$

$$\frac{dm_{g,t}}{d\bar{P}_{t+\tau}} \equiv \frac{1}{(1+\mu_{g,t+\tau})^{\tau}}, \tag{24}$$

$$\mu_{g,t+\tau} = \frac{1}{\left(\frac{dm_{g,t}}{d\bar{P}_{t+\tau}}\right)^{1/\tau}} - 1, \tag{25}$$

and

$$\mu_{g,t+\tau}^{annual} = (1+\mu_{g,t+\tau})^{1/6} - 1. \tag{26}$$

6.2 Valuing Government Payment Promises to the Unborn

Equations 27–31 present the analogue to equations 22–26 for those not yet born, i.e., for those for whom $\tau > g-1$:

$$EU_{-g,t} = \beta^g E_t[u\big(c_{1,t+g} - m_{-g,t} \prod_{i=t+1}^{t+g} (1+v_i)\big)] + \beta^{g+1} E_t[u(c_{2,t+g+1})] + \dots \tag{27}$$

$$+ \beta^\tau E_t[u\big(c_{\tau-g+1,t+\tau} + \tilde{\epsilon}_{t+\tau} \times \bar{P}_{t+\tau}\big)] + \dots + \beta^{g+9} E_t[u(c_{10,t+g+9})],$$

$$\frac{dm_{-g,t}}{d\bar{P}_{t+\tau}} = \beta^{\tau-g} \frac{E_t[u'(c_{\tau-g+1,t+\tau}) \times \tilde{\epsilon}_{t+\tau}]}{E_t[u'(c_{1,t+g}) \prod_{i=t+1}^{t+g} (1+v_i)]}, \tag{28}$$

where v_t equals $\bar{r}_{t-1}$ if the method of compensation is safe, and r_t otherwise.[22] The implied discount factors are computed and annualized as follows:

$$\frac{dm_{-g,t}}{d\bar{P}_{t+\tau}} = \frac{1}{(1+\mu_{-g,t+\tau})^\tau}, \tag{29}$$

$$\mu_{-g,t+\tau} = \frac{1}{\left(\frac{dm_{-g,t}}{d\bar{P}_{t+\tau}}\right)^{1/\tau}} - 1, \tag{30}$$

and

$$\mu^{annual}_{-g,t+\tau} = (1+\mu_{-g,t+\tau})^{1/6} - 1. \tag{31}$$

[22] Recall $\bar{r}_{t-1}$ is determined at time $t-1$ and realized at time t.

The difference in compensation is that $m_{-g,t}$, although determined at time t, can't be paid out to the unborn agent until the agent is born. In the meantime, $m_{-g,t}$ can be invested either at the sequence of ensuing "risk-free" returns or the ensuing risky returns on capital. As equation 28 makes clear, how $m_{-g,t}$ is invested will affect its size. This point is important. It means, for example, that the means by which future generations would be compensated for shutting down an ongoing government pension system will alter the calculated costs of doing so. The U.S. Social Security system's valuation of its unfunded liability is a case in point.[23] Social Security's actuaries calculate its unfunded liability annually on both 75-year and infinite-horizon bases.[24] The word "liability" references what is owed. To Social Security actuaries what is owed is what is needed, on average, to keep the system paying benefits. But to economists, what is owed is what is needed to fully compensate the creditor. This depends on the riskiness of Social Security's net benefit promises and, as just pointed out, on the risk arising due to the method used by Social Security to redeem its obligations to future generations.

7 Results

Turning to the results, we first illustrate the economy's transition path and then compare the volatility of the model's aggregate variables with the data. Next, we present the discount rates associated with pricing the safe and risky promises to the living and the unborn. We then consider the impact of the bond market on pricing. Finally, we use the data generated by our model to study how well specific Arbitrage Pricing Theory reduced forms can approximate the pricing of government obligations.

[23] Social Security references this as their *closed group liability*.

[24] See tables VIE1 and VIF1 in https://www.ssa.gov/OACT.

7.1 Illustrating the Economy's Path

Table 2 traces the economy from period 500 through period 520. Since each period represents roughly 6 years, 20 periods spans roughly 120 years. The table records, for each period, the safe and risky returns, the wage, the contemporaneous TFP shock (the lagged shocked is provided in the preceding years), the depreciation shock, the capital stock, and the distribution of cash on hand.

The first thing to note is that the economy exhibits a great deal of variability. The return to capital varies over the 20 years from a low of -6.72 percent to a high of 16.53 percent. The minimum safe return is 4.91 percent. The maximum is 9.28. And the wage, which is proportional to output given our Cobb-Douglas technology and fixed labor supply, is, at its maximum, 42.5 percent above its minimum. There is also very sizable variation in the capital stock, with the table's largest value exceeding its smallest by a factor of 2.49 (148.1 percent). This variation is driven by fluctuations in the depreciation rate. For example, during period 500, 81.4 percent of the previous stock of capital depreciates. On an annual basis, this is a 24.4 percent deprecation rate and explains why the rate of return in year 500 is -6.72 percent. It also explains why the capital stock falls almost in half between periods 500 and 501.

The major fluctuations in the rate of return as well as the wage lead to large differences from period to period in the age-distribution of cash on hand. Take, for example, the cash on hand share of age-3 agents. It varies from 5.7 percent to 10.2 percent. For 8-period year olds, the share varies from 11.7 percent to 15.6 percent.

Table 2 also indicates that when risky and riskless returns take on close to the same values in two particular periods, the configuration of state variables is quite similar. Periods 505 and 507 provide an example of this. So do periods 515 and 518.

Illustrating the Economy's Path

t	annual r (%)	annual rb (%)	w	z	delta	k	Cash on Hand as Percentage of Total Cash on Hand (%)									
							x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
500	-6.72	4.91	0.587	1.030	0.814	4.280	7.2	4.8	8.2	6.6	9.2	11.9	15.0	14.4	13.3	9.4
501	6.42	8.42	0.462	1.006	0.264	2.220	6.1	8.5	5.7	9.5	7.9	10.9	14.2	14.7	12.9	9.7
502	16.53	8.83	0.460	1.030	-0.731	2.053	4.7	7.6	10.2	7.3	11.5	10.0	13.4	13.7	12.8	8.9
503	8.39	6.75	0.514	1.014	-0.031	3.001	5.2	5.9	9.0	11.7	9.0	13.4	12.3	12.8	11.9	8.9
504	4.06	6.52	0.519	1.013	0.306	3.111	5.8	6.2	7.0	10.3	13.2	10.6	15.4	11.7	11.3	8.6
505	10.98	7.29	0.493	1.007	-0.238	2.696	4.9	7.0	7.5	8.4	11.9	15.1	12.8	14.5	10.1	7.8
506	4.36	6.37	0.525	1.016	0.275	3.196	5.7	5.9	8.1	8.7	9.7	13.4	16.8	12.0	12.4	7.3
507	11.93	7.00	0.502	1.008	-0.359	2.845	4.7	7.0	7.3	9.6	10.3	11.6	15.4	15.6	10.2	8.4
508	7.50	5.95	0.580	1.092	0.034	3.465	5.2	6.1	8.4	8.8	11.2	12.1	13.6	14.2	13.0	7.2
509	-0.33	5.89	0.534	0.990	0.527	3.633	6.2	6.0	6.9	9.4	9.9	12.6	14.0	13.0	12.6	9.5
510	6.93	7.72	0.448	0.942	0.129	2.471	5.6	7.2	7.0	8.0	10.9	11.7	14.9	13.7	11.7	9.1
511	3.56	8.02	0.440	0.949	0.428	2.292	6.3	6.3	8.2	7.9	9.2	12.6	13.8	14.5	12.4	8.7
512	9.69	9.28	0.412	0.952	0.023	1.856	5.7	7.3	7.3	9.5	9.3	10.8	14.8	13.4	12.9	8.9
513	12.19	9.02	0.427	0.973	-0.235	1.940	5.3	6.8	8.6	8.7	11.1	11.1	13.1	14.3	11.8	9.1
514	7.11	8.10	0.458	0.987	0.177	2.293	5.8	6.3	8.0	9.9	10.1	12.9	13.3	12.6	12.6	8.5
515	9.06	8.27	0.455	0.990	0.019	2.237	5.6	6.9	7.4	9.3	11.5	11.9	15.1	12.6	11.0	8.9
516	-0.77	7.95	0.475	1.014	0.736	2.371	7.2	6.3	7.7	8.2	10.3	12.9	13.7	14.3	11.2	8.3
517	13.54	9.92	0.428	1.016	-0.283	1.720	5.4	8.4	7.3	8.9	9.6	12.1	15.0	13.0	12.3	7.8
518	9.17	8.33	0.473	1.028	0.034	2.242	5.7	6.5	9.6	8.6	10.4	11.3	14.1	14.1	11.2	8.6
519	14.18	7.86	0.504	1.065	-0.505	2.448	4.8	7.1	8.0	11.3	10.4	12.3	13.5	13.2	11.9	7.6
520	0.06	6.16	0.558	1.057	0.562	3.404	6.5	5.7	8.0	9.0	12.4	11.8	14.1	12.6	11.4	8.6

Table 2: Tracing the economy through time, from period 500 to period 520. Cash on hand for each generation is expressed as a percent of total cash on hand. Rate of return on capital (r) and the risk-free rate (rb) are annualized and expressed in percent.

7.2 Comparing the Model's Volatility of Per Capital Output and Per Capita Consumption with the Data

Standard Deviation of Percent Deviations of Output and Consumption from Trend

Model/ Data	S.D. (%)
Output	
Model	9.947
Real Net National Product, 1929-2015	5.937
Aggregate Consumption	
Model	15.474
Real Personal Consumption Expenditures, 1929-2015	2.904

Table 3: Standard deviations of percent deviations from trend of U.S. per capita real net national product and U.S. per capita real personal consumption expenditures, 1929–2015 as well as standard deviations of percent deviations from the mean of output and aggregate consumption in the model. Data source: https://fred.stlouisfed.org. Reported NNP is converted to constant dollars using the GDP deflator. For each of the two annual data series, we first detrend using the Hodrick-Prescott filter, and then aggregate using 6-year rolling windows. The latter is done to allow for a fair comparison of the data with the model, in which each period represents six years. Using 6-year non-overlapping windows to aggregate the data yields similar results, namely a standard deviation of 6.977 percent for NNP and 2.800 percent for consumption expenditures.

Table 3 compares the variability of per capita output and per capita consumption from our model to their empirical counterparts. Following Prescott's (1986) procedure, we detrend these per capita series for the years 1929 through 2015 and form standard deviations of percent deviations from trend. Our model abstracts from growth, so we simply form the standard deviation of our model's percentage deviation of annual output from its mean. As the table shows, our model overstates HP-detrended actual per capita output variability by a factor of almost 2. It overstates the variability of actual HP-detrended per capita consumption by a factor of roughly 5. Hence, our finding (presented below) of small risk adjustment cannot be attributed to an understatement of HP-detrended output variability.

This said, HP-filtering removes significant lower-frequency fluctuations, which may reflect true economic uncertainty as opposed to expected cyclical variation. Certainly, other means of detrending the data, such as linear or quadratic detrending, produce far higher output variability. But whether that higher variability reflects unexpected variability is unclear.

Unfortunately, given our current simulation method, introducing far larger shocks to our model leads to approximation errors that preclude precise calculation of discount factors.

7.3 The Risk Premium Puzzle Revisited

Although our simulated economy's aggregate output, consumption, and factor prices are highly variable and its agents are quite risk averse, our model's risk premium is quite small. Calculated over 600 time periods, the mean risk-free rate is 6.9 percent, whereas the mean risky rate is 7.7 percent. Although this raises the Mehra and Prescott (1985) puzzle of why the observed (real world) risk premium is far higher than standard economic models predict, the reason our model's risk premium is small is that agents' don't view the risky asset as particularly risky.[25]

To see this, form the expectation, across time periods, of equation 16 for any age-group:

$$E\bar{r}_t = Er_{t+1} + E\Big[\frac{Cov_t(u'(c_{g+1,t+1}), r_{t+1})}{E_t u'(c_{g,t+1})}\Big], \forall g. \tag{32}$$

Since our estimate of the left hand side of equation 32 is 6.85 percent and our estimate of the first term on the right hand side is 7.67 percent, the risk premium is 0.82 percent. This risk premium is the same regardless of the generation, g, whose simulated data we use to form the risk premium. The risk premium's small size means that, regardless of which generation we consider, the generation does not, on average (across different time periods), expect its marginal utility of consumption next period to be strongly correlated to next period's risky rate of return. Were the risk premium precisely zero, the risky return would represent a zero-beta security. The return on a zero-beta security can be highly variable.

[25] As Hasanhodzic (2015) shows, the addition of sufficiently increasing borrowing costs to our model can generate an empirically realistic risk premium. Hasanhodzic and Kotlikoff (2018) replicate her results in an 80-period model. We omit borrowing costs here to simplify the analysis.

But if it's not correlated with an agent's consumption, it's viewed as safe. Thus, our finding of a small risk premium is synonymous with saying that our agents don't view the "risky" return as particularly risky.

7.4 Pricing Safe Promises to the Living

Each table in this section shows the discount rates that would be applied by agents at a given age who are a given number of periods away from receiving a sure payment in valuing that payment.

Annual Discount Factors for a Safe Asset for the Living
Model with Social Security
Low Risk-Free Rate, High Return on Capital Initial State

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.050	0.053	0.056	0.058	0.060	0.061	0.063	0.064	0.065
	2	0.049	0.052	0.055	0.057	0.060	0.061	0.063	0.064	
	3	0.051	0.053	0.056	0.058	0.061	0.062	0.064		
	4	0.048	0.052	0.055	0.057	0.060	0.062			
	5	0.049	0.052	0.055	0.058	0.060				
	6	0.050	0.053	0.056	0.058					
	7	0.050	0.053	0.056						
	8	0.051	0.054							
	9	0.049								

	Risk-Free Rate	Return on Capital
Current	0.050	0.108
Average	0.069	0.077

Table 4: Annual discount factors for a safe payment promised to the living. Initial state features a low risk-free rate and a high return on capital.

Tables 4 and 5 show the rates at which safe government payments are discounted, i.e., priced, in our model. These discount rates are defined in equations 22–26. Table 4 considers initial conditions, taken from the ergodic set. They produce an annualized low risk-free return on bonds, namely 5.0 percent and an annualized high risky return on capital, namely 10.8 percent.[26] Table 5 flips the relative sizes of prevailing (initial) risk-free and risky returns.

[26]No particular method was used to choose the state producing these returns. We simply scanned the

Annual Discount Factors for a Safe Asset for the Living
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.087	0.087	0.086	0.085	0.084	0.084	0.083	0.082	0.082
	2	0.087	0.086	0.086	0.085	0.084	0.083	0.082	0.082	
	3	0.085	0.084	0.084	0.082	0.082	0.081	0.080		
	4	0.087	0.087	0.086	0.085	0.085	0.084			
	5	0.088	0.087	0.087	0.086	0.085				
	6	0.088	0.087	0.087	0.086					
	7	0.089	0.088	0.088						
	8	0.088	0.088							
	9	0.087								

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 5: Annual discount factors for a safe payment promised to the living. Initial state features a high risk-free rate and a low return to capital.

The table's risk-free rate is 8.7 percent and its risky return is 4.1 percent. Again, the initial conditions producing these rates are in the ergodic set. A quick glance at Tables 4 and 5 shows that the discount rates needed to price safe payments are highly dependent on the prevailing one-period safe rate of return. Thus Table 4's discount rates are seemingly anchored to its 5.0 percent risk-free rate, whereas Table 5's discount rates strongly reflect its 8.7 percent discount rate.

As mentioned above, were our solution method free of approximation error, the values in the first column would each equal the prevailing risk-free rate. This isn't the case, but, again, the discrepancies, where they arise, are small and translate into even smaller percentage differences in the price of a one-period-ahead safe asset.

The two tables' results are interesting in a key respect. First, none of the discount rates, even many periods from the present, differ substantially from the initial-period risk-free return. This is the sense in which we describe these discount rates as anchored by

results until we found a low risk-free return and a high risky return that were significantly different in size. The same method was used to find the state underlying Table 5.

the prevailing one-period safe return. This seems both surprising and important. As we demonstrate via regressions presented in the next section (Generality of Findings), this finding holds regardless of initial conditions. I.e., there is nothing particularly special about Tables 4 and 5. Moreover, this finding that prevailing economic conditions are critical to pricing safe government obligations carries over to the pricing of risky obligations where the risk is strongly correlated with the wage.

The implied prescription to price safe and, potentially, most risky obligations based on current economic conditions is at strong odds with actual U.S. government practice. Take the Social Security system's Annual Trustees Report. Its annual reported liabilities are routinely discounted at a roughly 3.0 percent real return regardless of the economy's current condition, including its pricing of riskless securities as determined by real returns on Treasury Inflation Protected Securities.

Percentage Difference in Discount Factor Numerators and Denominators in Tables 4 and 5

		Numerator									Denominator
		Periods Till Benefit Received									
		1	2	3	4	5	6	7	8	9	
Current Age	1	0.701	0.432	0.260	0.142	0.058	-0.001	-0.038	-0.070	-0.108	1.100
	2	1.228	0.872	0.647	0.498	0.396	0.331	0.290	0.261		1.759
	3	1.698	1.304	1.062	0.903	0.798	0.733	0.700			2.263
	4	1.759	1.305	1.025	0.839	0.716	0.640				2.432
	5	1.814	1.353	1.067	0.876	0.751					2.500
	6	1.757	1.319	1.041	0.855						2.412
	7	1.596	1.173	0.903							2.226
	8	1.573	1.170								2.162
	9	2.446									3.271

Table 6: Percentage differences between the Table 5 (high risk-free rate, low return to capital) and Table 4 (low risk-free rate, high return to capital) numerators of the discount factor ratios and the percentage differences between the corresponding denominators.

What explains the strong influence of current economic conditions on the pricing of future safe payouts? The answer requires considering the denominator and numerator of the ratio determining the discount factor, namely $\frac{E_t u'(c_{g+\tau,t+\tau})}{u'(c_{g,t})}$. The larger is this ratio, the smaller is the discount factor. Note that the denominators in the discount-factor ratios are determined solely by current economic conditions. The numerators, in contrast, depends on expectations over future economic conditions. In Table 4, consumption at time t for all age groups is higher than in Table 5.[27] Hence, the denominator is lower in Table 4 than in Table 5 and, ignoring the numerator, all the Table 4 discount rates should, as observed, be lower than those in Table 5. Table 6 confirms this. It shows the percentage differences in the numerators and denominators in moving from Table 4 to Table 5.[28] As shown in the Table 6's last column, Table 5's denominator ranges from 110 percent to 327 percent larger than Table 4's denominator. This makes the discount ratio smaller and the discount rate higher in Table 5 compared to Table 4.

But Table 6 also shows that the numerators underlying Table 5's discount factors are generally higher than those underlying Table 4's discount factors. However, the percentage increases in the numerators are smaller than those of the denominators with smaller and smaller differences the larger is τ. I.e., current economic conditions, which determine the size of the denominator, are more important for long-run than for short-run discounting. Intuitively, the i.i.d. property of the depreciation shock, the dissipation through time of prior TFP shocks, and the ergodic nature of the underlying OLG model all point to similar expectations of the future regardless of the economy's current state. This means that the larger is τ, the closer are the numerators in Tables 4 and 5 and the more current conditions, again, captured by the denominator, determine the discount factor.

What determines whether the discount factors rise or fall with τ? The answer is how far the mean of the ergodic set is from the time-t state. Given the initial states underlying Tables

[27] In Table 4 consumption values are 0.424, 0.481, 0.500, 0.577, 0.618, 0.648, 0.693, 0.719, and 0.824 for ages 1 through 9, respectively. In Table 5, they are 0.331, 0.343, 0.337, 0.382, 0.407, 0.430, 0.469, 0.490, and 0.508, respectively.

[28] These percentages are calculated as Table 5 values minus Table 4 values divided by Table 4 values.

4 and 5, consumption levels rise, on average, through time from their time-t values. This means that the convergence of mean consumption is slower in Table 4 since consumption starts out higher. Consequently, the numerator in Table 4 falls more slowly than in Table 5, which means that the discount rate rises more rapidly in Table 4 than in Table 5 – exactly what we observe.

7.5 Generality of Findings

As mentioned, the anchoring of discount rates to the prevailing risk-free rate indicated in Tables 4 and 5 is not due to those tables' initial conditions (state vectors). Table 7 presents separate regressions, based on 600 observations, of the 9-period out discount rates of age-1 agents on either the risk-free rate prevailing when the agents are age 1 or their age-1 level of consumption. Both explanatory variables are highly significant. The fact that current consumption comes in so strongly affirms the point made above, namely that the discounting is being driven by current, not expected future conditions since expected future conditions are very similar regardless of the economy's current state.

7.6 Pricing Risky Promises to the Living

Before turning to the pricing results associated with risky promises, let us clarify how they should differ from the corresponding results for the safe promises presented above. Equation 33 decomposes the pricing of a risky security into safe and risky components:

$$\frac{E_t[u'(c_{t+\tau}) \times \epsilon_{t+\tau}]}{u'(c_t)} = \frac{E_t[u'(c_{t+\tau})]}{u'(c_t)} + \frac{Cov_t(u'(c_{t+\tau}), \epsilon_{t+\tau})}{u'(c_t)}, \tag{33}$$

where $E[\epsilon_{t+\tau}] = 1$.

The safe component, the first term on the equation's right hand side, is the price of a sure

Regressions of the Age 1, 9 Periods Out Discount Factors on the Current Risk-Free Rate (rb) or Current Consumption

Current risk-free rate (rb)		Current consumption	
Coefficients			
Intercept	0.328	**Intercept**	1.032
	(0.000)		(0.000)
Current rb	0.395	**Current cons**	-1.341
	(0.000)		(0.000)
Significance			
Adj. R^2	0.971	**Adj. R^2**	0.900
***p* (F)**	0.000	***p* (F)**	0.000
Obs.	600	**Obs.**	600

Table 7: Regressions of 9-period out discount factors of age 1 year olds on either the risk-free rate (rb) when the agent is age 1 or the agent's consumption at age 1. In the first regression, the t-statistic equals 224 for the intercept and 141 for the Current rb regressor. In the second regression, it equals 150 and -75 for the Intercept and Current cons, respectively. The corresponding p-values are zero to hundreds of decimal places.

promise, which we valued in Tables 4 and 5. The risky component reflects the covariance of consumption with the shock to the level of the average payment promised. Recall that $\epsilon_{t+\tau}$ is the ratio of the wage at time t to its mean. If consumption is higher (lower) when wages are higher (lower), the risky component will be negative, which, working through equations 23–25 and 33, implies a higher discount rate.

Annual Discount Factors for a Risky Asset for the Living
Model with Social Security
Low Risk-Free Rate, High Return on Capital Initial State

		Periods Till Benefit Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.050	0.055	0.058	0.060	0.062	0.063	0.064	0.066	0.066
	2	0.049	0.054	0.057	0.059	0.061	0.063	0.064	0.066	
	3	0.051	0.056	0.059	0.061	0.062	0.064	0.065		
	4	0.049	0.054	0.057	0.059	0.061	0.063			
	5	0.049	0.054	0.057	0.060	0.062				
	6	0.050	0.055	0.058	0.060					
	7	0.050	0.055	0.058						
	8	0.052	0.056							
	9	0.049								

	Risk-Free Rate	Return on Capital
Current	0.050	0.108
Average	0.069	0.077

Table 8: Annual discount factors for a risky payment to the living. Initial state features a low risk-free rate and a high return to capital.

Tables 8 and 9 price risky promises given the respective initial conditions underlying Tables 4 and 5. Tables 8 and 9 are identical to Tables 4 and 5 except that latter tables price promises to a sure unit of consumption whereas the former tables price promises to a unit of consumption multiplied by ϵ_t – the ratio of the realized wage to its average value.

The differences between values reported in Tables 8 and 4 as well as those between Tables 9 and 5 represent risk premiums. As is immediate from comparing the corresponding columns, the risk premiums are very close to zero. Consider, for example, the 8.2 percent rate at which an agent age 1 period discounts a safe promise 9 periods into the future in Table 5. If the promise is risky, in the manner specified, the discount rate is 8.3 percent (see

Table 9). Hence, the risk premium is only 0.1 percent.

Annual Discount Factors for a Risky Asset for the Living
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.088	0.088	0.087	0.086	0.086	0.085	0.084	0.083	0.083
	2	0.087	0.088	0.087	0.086	0.085	0.084	0.083	0.083	
	3	0.085	0.086	0.085	0.084	0.083	0.082	0.082		
	4	0.088	0.088	0.088	0.087	0.086	0.085			
	5	0.088	0.088	0.088	0.087	0.086				
	6	0.088	0.089	0.088	0.087					
	7	0.089	0.089	0.089						
	8	0.088	0.089							
	9	0.087								

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 9: Annual discount factors for pricing a risky payment to the living. The initial state features a high risk-free rate and a low return on capital.

There are two potential explanations for this finding. One is that there is very little unexpected variability in the wage, since the payment is proportional to the wage. The second is that this variability is not correlated with consumption.

Table 10 presents the coefficient of variation of the wage for each of the nine periods subsequent to the occurrence of the initial conditions introduced in Tables 4 and 5. These coefficients, starting from the initial conditions of Tables 4 and 8, rise from 3.1 percent one year out to 10.8 percent 9 years out. The corresponding coefficients starting from the initial conditions of Tables 5 and 9 are similar, rising from 3.1 percent to 10.2 percent. This is a rather small degree of variability.[29] Furthermore, not all of this variability reflects risk

[29]These coefficients of variation may suggest that our model has too little risk. But, according to Table 3, the model is producing more HP-filtered output and HP-filtered consumption variability than the actual economy. As for wages, the coefficient of variation of the HP-detrended median wage of full-time male workers reported by the St. Louis Federal Reserve is 0.027.[30] The corresponding coefficient of variation of wages in our model's simulated time series is 0.100. Hence, the model's wage variability is almost four times higher than the economy's HP-detrended wage variability. As indicated above, the robustness of our findings to alternative detrending methods is a topic for future research, but one that necessitates a simulation method that can produce highly precise results for even larger shocks than those considered here.

(unexpected changes). Some share of this variability was, by the nature of our model and its ergodic progress, expected well before the payment was made. This provided agents time to adjust their saving and, thereby, limit the impact of the wage "shock" on future consumption. This said, the relatively small wage variability goes a long way to explaining why wage-based risky government promises are valued as essentially safe.

Coefficient of Variation of the Wage for the Living

Initial State	Periods Till Payoff Received								
	1	2	3	4	5	6	7	8	9
Low Risk-Free Rate, High Return on Capital	0.031	0.075	0.093	0.102	0.108	0.110	0.110	0.109	0.108
High Risk-Free Rate, Low Return on Capital	0.031	0.064	0.079	0.088	0.095	0.098	0.099	0.100	0.102

Table 10: The coefficient of variation of the wage for each of the nine periods subsequent to the occurrence of the initial conditions characterized by the low (high) risk-free rate and high (low) return to capital.

Annual Discount Factors for a Risky Asset with Higher Payoff Volatility for the Living
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State

Current Age	Periods Till Payoff Received								
	1	2	3	4	5	6	7	8	9
1	0.091	0.101	0.102	0.101	0.100	0.098	0.095	0.093	0.092
2	0.090	0.100	0.102	0.101	0.099	0.097	0.095	0.093	
3	0.088	0.099	0.101	0.100	0.099	0.097	0.095		
4	0.090	0.101	0.102	0.100	0.099	0.097			
5	0.090	0.101	0.101	0.100	0.099				
6	0.090	0.100	0.101	0.100					
7	0.090	0.101	0.102						
8	0.090	0.100							
9	0.089								

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 11: Annual discount factors for pricing a high-volatility risky payment promised to the living. Here ϵ_t has the same mean but ten times the standard deviation of ϵ_t associated with the regular risky asset. The initial state is characterized by a high risk-free rate and a low return on capital.

Table 11 further investigates the risk premium associated with wage-based government payment promises. It repeats Table 9, but applies a mean-preserving spread to increase the variance of $\epsilon_{t+\tau}$ by a factor of 10.[31] Comparing the two tables shows that the long-term discount rates rise by roughly 100 basis points. Medium-term discount rates rise by close to 150 basis points. The message is that risk premiums associated with wage-linked government payments, particularly long-term risk premiums, are responsive to the level of risk. Still, they seem smaller than one might expect.

Illustrating the Comovement of Income Sources and Consumption

Age	z	delta	C	% Δ C	Wage	% Δ w	Stock Holding	Bond Holding	Stock Income	Bond Income	Total Income	Annual rb (%)	Annual r (%)	K
1	1.00	0.00	0.38	n/a	0.50	n/a	0.00	0.00	0.00	0.00	0.50	n/a	58.62	2.97
2	1.06	-0.27	0.46	20.52	0.54	6.12	0.23	-0.18	0.44	-0.27	0.71	6.69	11.11	3.00
3	0.95	0.83	0.34	-25.48	0.51	-4.84	0.34	-0.17	0.22	-0.24	0.50	5.88	-6.60	3.59
4	1.01	0.03	0.37	9.13	0.43	-14.95	0.18	-0.10	0.32	-0.17	0.59	9.42	10.28	1.81
5	1.01	0.13	0.40	7.72	0.45	3.73	0.23	-0.08	0.38	-0.14	0.69	8.96	8.61	2.01
6	0.96	0.14	0.42	4.52	0.43	-3.43	0.27	-0.05	0.43	-0.08	0.79	8.71	7.88	2.10
7	0.98	-0.04	0.46	10.60	0.44	0.78	0.30	0.01	0.53	0.01	0.98	8.66	9.96	2.07
8	1.00	-0.63	0.59	27.86	0.46	5.87	0.29	0.16	0.69	0.25	0.94	8.17	15.20	2.27
9	1.12	0.72	0.52	-11.46	0.58	24.57	0.32	0.19	0.29	0.27	0.56	6.37	-1.71	3.19
10	1.12	0.12	0.58	10.19	0.52	-9.18	0.14	0.09	0.24	0.15	0.39	8.34	8.71	2.33

Table 12: Examples of the comovement of income sources and consumption in the presence of the bond market. The 10 rows follow an agent over her life cycle, from age 1 through 10.

Why are changes in wage rates so poorly correlated with changes in the marginal utility of consumption? Table 12 provides an answer. It follows, through her life cycle, an agent born in the mean initial state, i.e., in a state determined by the mean of the state variables generated in our 600-period simulation. The 10 rows start with the agent at age 1 and follow her through age 10.[32]

As the table shows, consumption can move in opposite directions to wages. This occurs in the table in 4 out of the 9 periods.[33] Take, for example, the 14.95 percent decline between

[31] To be specific, we use $10(\epsilon_{t+\tau} - E[\epsilon_{t+\tau}]) + 1$ where $E[\epsilon_{t+\tau}] = 1$.
[32] The shocks used to form the table were drawn at random.
[33] There is no consumption change for 1 period olds.

ages 3 and 4. Rather than falling, consumption rises by 9.13 percent. Or consider the 24.6 percent rise in wages when the agent is age 9. Her consumption in that period falls by 11.5 percent. The explanation for these cases is that depreciation shocks dramatically impact asset income. This means total income can fall even when wage income rises and vice versa. Moreover, for older workers, increases in wage income have no direct impact on total income since agents are retired in periods 8, 9, and 10.

Table 13 reports the correlation between the percentage change in the wage and percentage change in consumption for each age group. To compute the correlation, we simulate the model 10,000 times always starting from the same initial condition, which corresponds to the non-stochastic steady state. The 10 columns follow an agent over her life cycle, from age 1 through 10. For each age group, correlation is computed across simulations. The largest of the table's correlation coefficients is the 0.220 age-2 coefficient. Seven of the nine coefficients are negative and nine of the ten are very small. This table suggests that securities linked to wages, like securities linked to the return to capital are, effectively, zero beta securities and, consequently, are not riskier than safe securities.

Correlation Between Percent Change in Wage and Percent Change in Consumption

Age	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Correlation	0.220	-0.012	-0.009	0.013	-0.023	-0.002	-0.021	-0.005	-0.020

Table 13: Correlation between the percentage change in the wage and percentage change in consumption for each age group.

Annual Discount Factors for a Government Option for the Living
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.227	0.163	0.142	0.130	0.122	0.115	0.110	0.106	0.103
	2	0.226	0.163	0.142	0.129	0.121	0.115	0.109	0.106	
	3	0.223	0.161	0.141	0.128	0.120	0.114	0.109		
	4	0.225	0.163	0.142	0.129	0.121	0.115			
	5	0.225	0.163	0.141	0.129	0.121				
	6	0.224	0.162	0.141	0.129					
	7	0.223	0.162	0.142						
	8	0.223	0.162							
	9	0.223								

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 14: Annual discount factors for a government option provided to the living. The payoff of the option equals that of the risky asset if $\epsilon_t > 1$ and zero otherwise. The initial state features a high risk-free rate and a low return on capital.

7.7 Pricing a Government Option to Make Payments to the Living

Table 14 values a government option to make wage-tied payments to the agent, but only if wages are above their mean. Otherwise, the payment is set to zero. These results illustrate the capacity of the model to price all types of securities. The table considers the initial conditions from Table 5, i.e., a high 8.7 percent risk-free rate and a low 4.1 percent return on capital. This table's 1-period-from-payoff discount rates (i.e., first column) are almost three times as large as the rates presented in Table 9. These discount rates decline by over one half as the number of periods to payoff rises from 1 to 9.

Clearly, the risk premiums decline the farther out the option. For the 1-period old agent, the risk premium, measured on an annual basis, is roughly 22.5 percent less 8.7 percent, the prevailing risk-free rate, or 13.8 percent. For a 9-period out payoff, the risk premium is is 10.3 percent less 8.7 percent or just 1.6 percent. On the other hand, this risk premium compounds, reducing in roughly half the price a current one-period-old agent would pay for the option as compared to the sure payoff.[34]

Why does the option's discount rate fall with the time it's paid? The answer is the probability it will be in the money. In the initial state underlying the table, the wage is below its mean and, given serial correlation in the TFP shock, there is a better than even chance that it will remain below its mean in the short run. High short-run discount rates means a lower price for the option. Over time, the impact of the initial state dissipates and the probability that the wage will exceed its mean rises. This is why longer-duration options are more valuable (have a smaller discount rate).

7.8 Pricing Safe and Risky Payments to the Unborn

Annual Discount Factors for a Safe Asset for the Unborn
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State
Risky Method of Compensation

		Age at the Receipt of the Asset								
		1	2	3	4	5	6	7	8	9
Current Age	-1	0.089	0.087	0.085	0.084	0.083	0.083	0.082	0.081	0.081
	-2	0.088	0.086	0.085	0.084	0.083	0.082	0.081	0.081	0.081
	-3	0.087	0.085	0.084	0.083	0.082	0.082	0.081	0.081	0.080
	-4	0.086	0.085	0.083	0.083	0.082	0.081	0.081	0.080	0.080
	-5	0.085	0.084	0.083	0.082	0.081	0.081	0.080	0.080	0.080
						...				
	-18	0.080	0.079	0.079	0.079	0.079	0.078	0.078	0.078	0.078
	-19	0.079	0.079	0.079	0.079	0.078	0.078	0.078	0.078	0.078
	-20	0.079	0.079	0.079	0.079	0.078	0.078	0.078	0.078	0.078

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 15: Annual discount factors for pricing a safe payment promised to the unborn. The initial state features a high risk-free rate and a low return on capital.

[34]The amount $\frac{1}{1.082^{54}}$ is the price of a safe payment paying off in 9 periods as perceived by an age-1 period agent (see column 9 in Table 5). The amount $\frac{1}{1.103^{54}}$ (see column 9 in Table 14) is the corresponding price of the risky option. The number 54 is based on our assumption that each period stands for 6 years and we are considering 9 times 6 periods into the future.

Annual Discount Factors for a Risky Asset for the Unborn
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State
Risky Method of Compensation

		Age at the Receipt of the Asset								
		1	2	3	4	5	6	7	8	9
Current Age	-1	0.089	0.088	0.086	0.086	0.085	0.084	0.083	0.082	0.082
	-2	0.089	0.087	0.086	0.085	0.084	0.083	0.082	0.082	0.081
	-3	0.087	0.086	0.085	0.084	0.083	0.082	0.082	0.081	0.081
	-4	0.087	0.085	0.084	0.083	0.083	0.082	0.081	0.081	0.081
	-5	0.086	0.084	0.084	0.083	0.082	0.082	0.081	0.081	0.080
						...				
	-18	0.080	0.080	0.079	0.079	0.079	0.079	0.079	0.078	0.078
	-19	0.080	0.079	0.079	0.079	0.079	0.079	0.079	0.078	0.078
	-20	0.079	0.079	0.079	0.079	0.079	0.079	0.078	0.078	0.078

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 16: Annual discount factors for pricing a risky payment promised to the unborn. The initial state features a high risk-free rate and a low return on capital.

Tables 15 and 16 price government payments to future generations. Table 15 and considers safe payments and Table 16 considers risky payments. As indicated in equations 27 and 28, these tables consider the government compensating the unborn by holding aside a given amount of resources, denoted by $m_{-g,t}$, which is invested at the prevailing rates of return to capital through the period of birth of the future generation in question. Since the findings are qualitatively very similar regardless of initial conditions, we present results just for the initial conditions of Table 5.

Unlike in the prior tables, the discount rates of future generations vary more distinctly by the age of the agent. For example, agents who are age -1 period discount a safe payment 1 period from now at an 8.9 percent rate, whereas agents age -20 discount at a 7.9 percent rate.[35] We also find that making the government payment risky (proportional to the wage) makes virtually no difference to future generation's pricing of these promises.

[35]Under the alternative initial conditions, one-year out discount rates are again close to the prevailing discount rates for younger future agents. But they can be higher or lower for older future generations depending on the initial state.

7.8.1 Impact of Method of Compensation on Pricing Payments to the Unborn

Annual Discount Factors for a Safe Asset for the Unborn
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State
Safe Method of Compensation

		Age at the Receipt of the Asset								
		1	2	3	4	5	6	7	8	9
Current Age	-1	0.086	0.085	0.084	0.083	0.083	0.082	0.081	0.081	0.080
	-2	0.085	0.084	0.083	0.082	0.082	0.081	0.081	0.080	0.080
	-3	0.084	0.083	0.082	0.081	0.081	0.080	0.080	0.080	0.079
	-4	0.083	0.082	0.081	0.081	0.080	0.080	0.080	0.079	0.079
	-5	0.083	0.081	0.081	0.080	0.080	0.079	0.079	0.079	0.079
						...				
	-18	0.077	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076
	-19	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076
	-20	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 17: Annual discount factors for pricing a safe payment promised to the unborn. These safe government payment promises to be made at different future dates are reinvested at the risk-free rate. The initial state features a high risk-free rate and a low return on capital.

Like Tables 15 and 16, Tables 17 and 18 present the rates at which unborn (future) generations would discount safe government payment promises to be made at different future dates. However, while the former pair of tables consider a risky method of compensation under which the amount m is reinvested at the risky return to capital, the later pair of tables consider a "safe" method of compensation under which the amount m is reinvested at the sequence of safe rates of return.

Comparing Tables 15 and 17, which price safe promises, we can see that the discount rates are uniformly lower when the method of compensation is safe. For example, agents who are age -1 period discount a safe promise to be received when they are born at an 8.9 percent rate when the method of compensation is risky, whereas they discount at an 8.6 percent rate when the method of compensation is safe. Similarly, agents who are -20 periods old discount at a 7.9 percent rate when the method of compensation is risky, and at a 7.6

Annual Discount Factors for a Risky Asset for the Unborn
Model with Social Security
High Risk-Free Rate, Low Return on Capital Initial State
Safe Method of Compensation

		Age at the Receipt of the Payoff								
		1	2	3	4	5	6	7	8	9
Current Age	-1	0.087	0.086	0.085	0.084	0.084	0.083	0.082	0.082	0.081
	-2	0.086	0.085	0.084	0.083	0.082	0.082	0.081	0.081	0.081
	-3	0.085	0.084	0.083	0.082	0.082	0.081	0.081	0.080	0.080
	-4	0.084	0.083	0.082	0.082	0.081	0.081	0.080	0.080	0.080
	-5	0.083	0.082	0.082	0.081	0.081	0.080	0.080	0.079	0.079
						...				
	-18	0.077	0.077	0.076	0.076	0.076	0.076	0.076	0.076	0.076
	-19	0.077	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076
	-20	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076	0.076

	Risk-Free Rate	Return on Capital
Current	0.087	0.041
Average	0.069	0.077

Table 18: Annual discount factors for pricing a risky payment promised to the unborn. These risky government payment promises to be made at different future dates are reinvested at the risk-free rate. The initial state features a high risk-free rate and a low return on capital.

percent rate when it is safe. Comparing Tables 16 and 18 shows that the same conclusions hold when pricing risky promises.

Tables 19 and 20 present a case where the discrepancy between discount rates for different methods of compensation is particularly pronounced. These tables price a safe promise to the unborn in a model without Social Security and starting from a typical initial state. Now, the discount rate declines from 4.6 percent with risky method of compensation to 4.2 percent with safe method of compensations for a -1 period old agent, and from 5.4 percent with risky method of compensation to 4.8 percent with safe method of compensation for a -20 periods old agent. Interestingly, 5.4 percent is close to the average return on capital of 5.6 percent, and 4.8 percent is close to the average risk-free rate of 4.6 percent. This result makes sense. Since starting from a typical state there is no transition period during which the economy will revert to some average state of nature, the rates of return that will prevail in each of the ensuing 20 periods (until the -20 period old agent is born) will average out to their long-run

Annual Discount Factors for a Safe Asset for the Unborn
Model without Social Security and Typical Initial State
Risky Method of Compensation

	Age at the Receipt of the Asset								
Current Age	1	2	3	4	5	6	7	8	9
-1	0.046	0.046	0.046	0.047	0.047	0.048	0.048	0.049	0.049
-2	0.048	0.047	0.047	0.048	0.048	0.049	0.049	0.050	0.050
-3	0.049	0.048	0.048	0.049	0.049	0.049	0.050	0.050	0.050
-4	0.050	0.049	0.049	0.049	0.050	0.050	0.050	0.050	0.051
-5	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.051	0.051
					...				
-18	0.054	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
-19	0.054	0.054	0.053	0.053	0.053	0.053	0.053	0.053	0.053
-20	0.054	0.054	0.054	0.053	0.053	0.053	0.053	0.053	0.053

	Risk-Free Rate	Return on Capital
Current	0.040	0.057
Average	0.046	0.056

Table 19: Annual discount factors for pricing a safe payment promised to the unborn. These safe government payment promises to be made at different future dates are reinvested at the return to capital ("risky method of compensation"). The initial state corresponds to a typical state of the economy (the middle of the ergodic distribution).

Annual Discount Factors for a Safe Asset for the Unborn
Model without Social Security and Typical Initial State
Safe Method of Compensation

	Age at the Receipt of the Asset								
Current Age	1	2	3	4	5	6	7	8	9
-1	0.042	0.043	0.044	0.045	0.046	0.046	0.047	0.048	0.048
-2	0.043	0.043	0.044	0.045	0.046	0.047	0.047	0.048	0.048
-3	0.043	0.044	0.045	0.046	0.046	0.047	0.048	0.048	0.048
-4	0.045	0.045	0.046	0.046	0.047	0.047	0.048	0.048	0.049
-5	0.045	0.046	0.046	0.047	0.047	0.048	0.048	0.048	0.049
					...				
-18	0.048	0.048	0.048	0.048	0.049	0.049	0.049	0.049	0.049
-19	0.048	0.048	0.048	0.048	0.049	0.049	0.049	0.049	0.049
-20	0.048	0.048	0.048	0.048	0.049	0.049	0.049	0.049	0.049

	Risk-Free Rate	Return on Capital
Current	0.040	0.057
Average	0.046	0.056

Table 20: Annual discount factors for pricing a safe payment promised to the unborn. These safe government payment promises to be made at different future dates are reinvested at the risk-free rate ("safe method of compensation"). The initial state corresponds to a typical state of the economy (the middle of the ergodic distribution).

average values. Hence, it is those long-run average values that will matter most for pricing.

7.9 Impact of the Bond Market on Pricing

To assess the impact of the bond market on discount rates, we re-simulated the model without that market.[36] Interestingly, the bond market doesn't materially impact the economy's aggregate variables. This is the same finding reported in Hasanhodzic and Kotlikoff (2013, 2018) and Hasanhodzic (2015). Indeed, average returns to capital with and without bonds are identical to three decimal places.

Illustrating the Comovement of Income Sources and Consumption
No Bond Market

Age	z	delta	C	% Δ C	Wage	% Δ w	Stock Holding	Bond Holding	Stock Income	Bond Income	Total Income	Annual rb (%)	Annual r (%)	K
1	1.00	0.00	0.37	n/a	0.50	n/a	0.00	n/a	0.00	n/a	n/a	0.00	7.99	2.97
2	1.06	-0.27	0.43	14.82	0.54	6.20	0.06	n/a	0.10	n/a	n/a	0.00	11.10	3.01
3	0.95	0.83	0.37	-13.05	0.51	-5.31	0.13	n/a	0.09	n/a	n/a	0.00	-6.51	3.54
4	1.01	0.03	0.41	10.71	0.44	-13.15	0.15	n/a	0.26	n/a	n/a	0.00	9.99	1.91
5	1.01	0.13	0.44	7.68	0.46	5.17	0.22	n/a	0.35	n/a	n/a	0.00	8.09	2.21
6	0.96	0.14	0.46	3.86	0.45	-2.59	0.30	n/a	0.46	n/a	n/a	0.00	7.25	2.37
7	0.98	-0.04	0.51	10.20	0.46	0.89	0.38	n/a	0.65	n/a	n/a	0.00	9.35	2.34
8	1.00	-0.63	0.71	40.26	0.48	5.46	0.53	n/a	1.21	n/a	n/a	0.00	14.78	2.53
9	1.12	0.72	0.54	-24.74	0.59	23.02	0.66	n/a	0.58	n/a	n/a	0.00	-2.26	3.43
10	1.12	0.12	0.59	9.24	0.53	-9.86	0.25	n/a	0.40	n/a	n/a	0.00	8.43	2.45

Table 21: Examples of the comovement of income sources and consumption without the bond market. The 10 rows follow an agent over her life cycle, from age 1 through 10.

The bond market does, however, help contemporaneous generations share risk. To see this, compare Table 21, which has no bond market, with Table 12, which does. Both tables follow a new born agent starting in the same initial state (the economy's non-stochastic steady state where period-specific shocks are the same across the next nine periods. As

[36] In so doing, we simply omit the choice of $\alpha_{g,t}$ as well as the constraint that bond holdings sum to zero.

suggested by the negative (positive) bond holdings of the young (old), younger agents use the bond market to insure older agents. I.e., younger agents sell bonds to older agents and invest the proceeds in stocks. This makes the consumption of the young riskier and that of the old less risky. As Table 22 confirms, this is not a special example. Coefficients of variation of consumption by age are larger for younger agents and smaller for older agents in the presence of the bond market.

Mean and Coefficient of Variation (CV) of Consumption by Age

Age	1	2	3	4	5	6	7	8	9	10
Bond Market										
Mean	0.376	0.402	0.430	0.459	0.491	0.524	0.559	0.597	0.637	0.680
CV	0.099	0.123	0.141	0.155	0.164	0.169	0.173	0.179	0.182	0.184
No Bond Market										
Mean	0.369	0.395	0.423	0.452	0.484	0.517	0.554	0.599	0.650	0.706
CV	0.072	0.071	0.078	0.090	0.106	0.125	0.148	0.199	0.242	0.279

Table 22: Mean and coefficient of variation (CV) of consumption by age. The statistics for 1-year olds were computed by pooling all of the 1-year olds across 600 periods of the simulation, and similarly for all other age groups.

The young clearly face more consumption variability and the old less consumption variability when the young and old can share risk via the bond market. This shifting of risk reminds us that the bond market doesn't eliminate macro risk, it just shares it. Consequently, we shouldn't expect the bond market to uniformly impact discount rates regardless of age.

Tables 23 and 24 display discount rates for safe and risky assets but assuming no bond market. Both Tables are based on Table 4's state vector. The impact of the bond market can be seen by comparing the first columns in Tables 4 and 23, both of which price a safe government payment one period out. In the former table, each of the discount rates for a safe payment one year out is very close to 5.0 percent, the prevailing risk free rate. In the later table, the rates range from 5.8 to 4.0 percent – a sizable difference. The rates are, with one exception, higher for younger agents and lower for older agents. This pattern of higher discount rates for the young and lower rates for the old extends to longer-dated safe government payments.

Annual Discount Factors for a Safe Asset for the Living
Model with Social Security and No Bond Market
Same Initial State as in the Case of Low Risk-Free Rate, High Return on Capital

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.058	0.059	0.061	0.062	0.063	0.064	0.063	0.064	0.063
	2	0.053	0.055	0.057	0.059	0.060	0.060	0.061	0.061	
	3	0.053	0.055	0.057	0.058	0.058	0.059	0.059		
	4	0.049	0.051	0.054	0.054	0.055	0.056			
	5	0.047	0.050	0.051	0.051	0.053				
	6	0.046	0.046	0.049	0.050					
	7	0.040	0.043	0.046						
	8	0.043	0.043							
	9	0.042								

	Risk-Free Rate	Return on Capital
Current	–	0.108
Average	–	0.077

Table 23: Annual discount factors for pricing a safe payment to the living assuming no bond market. Initial state features the same low risk-free rate and high return to capital as in the corresponding table with the bond market.

These results are intuitive. With the short-term bond market, older generations limit their risk by buying bonds from younger generations. This means that absent the ability to buy and sell bonds, the young value a safe asset less than the elderly, i.e., they discount a safe payment at a higher rate.[37]

Next consider Table 24, which prices risky government payments provided to the living in the absence of a bond market. The discount rates are strikingly similar to those in Table 23. Hence, once again we find no risk premium for a risky compared to a safe payment. This

[37]Younger generations face relatively more risk from TFP shocks than older ones whose principal is insulated from the shocks. But this principal is directly impacted by depreciation shocks. Such shocks also impact workers via lower wages, but on balance the risks facing the elderly from both shocks appear to outweigh those facing the young, who, of course, have more periods over which to recoup losses.

Annual Discount Factors for a Risky Asset for the Living
Model with Social Security and No Bond Market
Same Initial State as in the Case of Low Risk-Free Rate, High Return on Capital

		Periods Till Payoff Received								
		1	2	3	4	5	6	7	8	9
Current Age	1	0.058	0.060	0.062	0.063	0.064	0.064	0.064	0.065	0.064
	2	0.053	0.056	0.059	0.060	0.061	0.061	0.062	0.062	
	3	0.053	0.056	0.058	0.059	0.060	0.060	0.061		
	4	0.049	0.053	0.055	0.055	0.057	0.058			
	5	0.048	0.051	0.052	0.054	0.055				
	6	0.047	0.048	0.051	0.053					
	7	0.040	0.046	0.050						
	8	0.043	0.047							
	9	0.042								

	Risk-Free Rate	Return on Capital
Current	–	0.108
Average	–	0.077

Table 24: Annual discount factors for pricing a risky payment to the living with no bond market. Initial state features the same high risk-free rate and low return to capital as in the corresponding table with the bond market.

is true even for end-of-period 1 payments. This reflects the lack of consumption-correlated risk of the wage-based payment security. The other key point about our results with no bond market is that the discount rates, as in the case with a bond market, are largely anchored by what the short-term bond rate would be in the presence of a bond market.

8 Using Our Structural Model To Evaluate Arbitrage Pricing

Data generated by our model can be used to determine how well specific APT reduced-forms can approximate the correct pricing of government obligations. To this end, we present, in Table 25, regressions of the growth rate of wages on contemporaneous safe and risky returns, on returns lagged by one period, and on contemporaneous plus multiple lagged returns.

Note that using either multiple lags plus contemporaneous returns or one-period lags

Regressions of Growth Rate of Wages on Contemporaneous and Lagged Safe Returns (rb) and Risky Returns (r)

Contemporanous plus four period lagged safe and risky returns		One period lagged safe and risky returns		Contemporanous safe and risky returns	
Coefficients					
Intercept	-0.127***	**Intercept**	-0.108***	**Intercept**	0.096***
	(0.000)		(0.000)		(0.000)
r(t)	0.037**	**r(t-1)**	0.177***	**r(t)**	-0.022***
	(0.053)		(0.000)		(0.009)
r(t-1)	0.172***	**rb(t-1)**	0.010	**rb(t)**	-0.157***
	(0.000)		(0.377)		(0.000)
r(t-2)	-0.025				
	(0.184)				
r(t-3)	0.001				
	(0.954)				
r(t-4)	0.012***				
	(0.006)				
rb(t)	0.133*				
	(0.092)				
rb(t-1)	-0.149				
	(0.216)				
rb(t-2)	-0.095				
	(0.427)				
rb(t-3)	0.139				
	(0.237)				
rb(t-4)	-0.003				
	(0.973)				
Significance					
Adj. R^2	0.761	**Adj. R^2**	0.758	**Adj. R^2**	0.072
***p* (F)**	0.000	***p* (F)**	0.000	***p* (F)**	0.000
Obs.	596	**Obs.**	596	**Obs.**	596

Table 25: Regressions of the growth rate of wages on contemporaneous and lagged returns. Statistical significance is denoted by stars at the 10% (*), 5% (**) and 1% (***) levels, respectively.

produces high values of R^2 – above 0.75. In contrast, the R^2 in the regression with just contemporaneous returns is only 0.072. On the other hand, each regression has highly significant variables.[38]

Annual Discount Rates Based on APT

	Periods Till Payoff Received								
	1	2	3	4	5	6	7	8	9
	Contemporaneous Returns								
High Rb, Low R; Prevailing Rb	0.084	0.084	0.084	0.084	0.084	0.084	0.084	0.084	0.084
High Rb, Low R; Average Rb	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072
Low Rb, High R; Prevailing Rb	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.057	0.057
Low Rb, High R; Average Rb	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072
	Lagged Returns								
High Rb, Low R; Prevailing Rb	0.087	0.085	0.085	0.084	0.079	0.075	0.073	0.072	0.071
High Rb, Low R; Average Rb	0.070	0.071	0.071	0.071	0.067	0.064	0.062	0.061	0.060
Low Rb, High R; Prevailing Rb	0.054	0.056	0.056	0.057	0.053	0.051	0.050	0.049	0.048
Low Rb, High R; Average Rb	0.074	0.073	0.072	0.072	0.067	0.065	0.063	0.061	0.060

Table 26: Annual discount-rate results from using the APT pricing formulas of Blocker, Kotlikoff, and Ross (2019) to value wage-based payment promises for the same initial high risk-free, low risky rate and low risk-free, high risky rate cases considered previously in the paper. The APT pricing is based on wage-growth rate regressions using either contemporaneous returns or one-period lagged returns. Valuation results from using both the prevailing risk-free rate and the average risk-free rate are presented.

Blocker, Ross and Kotlikoff (2006) present formulas for APT pricing based on wage-growth rate regressions using either a) contemporaneous returns or b) one-period lagged returns. In the case of contemporaneous returns, they posit the following relationship be-

[38]The first regression with 4 lags plus contemporaneous returns provides only limited support for the Lucas and Zeldes (2006) and Geanakoplos and Zeldes (2010,2011) view that returns and wages are strongly correlated over the long term, but not over the short term. Of course, these authors are making observations about actual data, not data simulated from a highly stylized model. They are also focused on correlations in annual data, not correlations in what amounts to roughly six-year data. The basis for the correlation between current returns and wage growth in our model is simply a standard production function in which both labor and capital can be immediately adjusted. Modifying our framework to incorporate capital adjustment costs would move our model closer to the Lucas-Tree model that Geanakoplos and Zeldes (2011) posit to support their empirical findings. Table 25 includes a regression of wage growth at time t on a constant and current and lagged risky returns, all of which are significant. Based on their model, Geanakoplos and Zeldes (2011) would view the significance of the lagged regressors as reflecting early news about future productivity shocks. But in our model, agents don't learn about shocks until they occur. Instead, the ability, in our model, of past returns to predict future wage growth today reflects the economy's ergodic process. It also tells us that agents can infer much of what's coming with respect to wage-based government payments far before those payments are made.

tween annual wage growth rates and current returns on market securities.

$$\frac{w_{t+1}}{w_t} = 1 + \alpha + \sum_i \beta_i f_{i,t} + \epsilon_t, \tag{34}$$

where w_t is the economy's average wage at time t, α is a constant, $f_{i,t}$ is the return to asset i at time t, and the β_i's are regression coefficients.

To value an obligation that's proportional to the future level of w_{t+1}, they determine the amount, $A_{i,t}$, one would need to invest in risky asset i and the amount B_t one would need to invest in a safe asset yielding an assumed fixed safe return, $\bar{r}$, to replicate w_{t+1}, apart from idiosyncratic risk. This entails setting $A_{i,t} = \beta_i$ and $B_t = \frac{1+\alpha-\sum_i \beta_i}{1+\bar{r}}$. The value, V_T, of a wage growth security that pays out w_T in expected value is given by

$$V_{t,T} = w_0 \Big(\frac{1 + \alpha + \bar{r} \sum_i \beta_i}{1 + \bar{r}} \Big)^T. \tag{35}$$

In the case that wage growth is related to one-period lagged returns, the analogous formula is

$$V_{t,T} = w_0 \Big(\frac{1 + \alpha + \sum_i \beta_i f'_{i,t-1}}{1 + \bar{r}} \Big) \Big(\frac{1 + \alpha + \bar{r} \sum_i \beta_i}{1 + \bar{r}} \Big)^{T-1}. \tag{36}$$

These two formulas differ in important ways from our consumption-asset pricing. Equation 35 assumes that the safe rate of return, $\bar{r}$, is constant through time. It also provides the same pricing no mater the age of the recipient of the government's promise to make a payment that's proportional to the wage.[39] Finally, although we implement the formula using either the prevailing or average safe rate of return, the formula itself doesn't depend on time since, again, $\bar{r}$ is assumed time-invariant. Equation 36 does depend on time insofar as it includes lagged returns. But as with equation 35, it assumes a time-invariant interest rate.

Blocker et al. (2006) also point out that APT pricing becomes highly complex if one prices wage growth based on asset returns extending over many lags. As mentioned, Lucas and Zeldes (2006) and Geanakoplos and Zeldes (2010, 2011) surmount this APT pricing complexity by assuming a tightly structured reduced form relating wage growth to past security returns.[40]

Table 26 presents eight sets of annualized discount-rate results from using these APT pricing formulas to value the same wage-based risky government payment promises considered in Tables 8 and 9. Table 26's discount rates are directly comparable to their counterparts in those tables. The first four rows in the table are based on the contemporaneous-returns regression. Rows 1 and 2 consider the initial high risk-free, low risky returns case. Rows 3 and 4 consider the initial low risk-free, high risky rate case. Rows 1 and 3 use the prevailing risk-free rate in the valuation formula. Rows 2 and 4 follow Social Security actuaries' practice of using the average risk-free rate. Row 5 through 8 are the counterparts of rows 1 through 4 except that they use the one-period lagged return formula.

The first thing to point out is that Table 26's discount rates based on contemporaneous returns are close, but not identical to those based on returns lagged one period. This suggests the importance of correctly specifying the reduced form of the arbitrage pricing relationship. The second point is that the APT pricing is fairly similar to that reported in Tables 8 and 9 *provided one implement the formulas using prevailing, not average safe*

[39] The value of such a promise is simply proportional to V_T.

[40] Benzoni et al. (2007) test for cointegration between labor and capital income by running an augmented Dickey-Fuller (ADF) test to check whether the variable representing the difference between log-aggregate labor income and log-dividend is stationary. In our model, the dividend in any period is given by $r_t K_t$, and the difference between the log of wages and the log of dividends is always a constant ($log(\frac{w_t}{r_t K_t}) = \frac{1-\alpha}{10\alpha}$).

rates of return. The upshot here is that which APT formulation one uses matters, but the particular implementation of the formula matters even more. If one uses, as do the Social Security actuaries, average rather than prevailing safe rates of return in forming estimates of government liabilities, one will produce very different pricing. This is particularly clear by comparing rows 3 and 4 and rows 7 and 8 in Table 26.

9 Conclusion

The proper way to value government commitments when markets are incomplete is a long-standing, fundamental, yet unresolved question in economics. The reduced-form approach, which relies on arbitrage pricing, requires strong assumptions about the availability of spanning securities (or implicit factors) and the manner of spanning. The alternative approach, taken here, is to posit, calibrate, and solve a structural model and use consumption-asset pricing (compensating variation) to price government obligations.

In the past, the curse of dimensionality limited economists' ability to solve what is arguably one of the most realistic structural frameworks – a large-scale OLG model with macro shocks. But computational breakthroughs have made solving such models eminently feasible. This paper provides an example. It solves a 10-period OLG model with large, indeed overly large productivity and depreciation shocks.

We use our model to price safe and risky, short- and long-term government payment promises made to both current and future generations. As in Mehra and Prescott (1985), our model generates a small risk premium even though the risky return in our model is highly variable. The explanation is that the risky return is essentially a zero-beta security, i.e., a security that can have a very variable return but still be priced as a safe asset because its risk is uncorrelated with the market (in our case, consumption).

Our main finding is that prevailing, rather than average (across-time) short-term safe real bond rates play a crucial role in determining the pricing of short-, medium-, and even long-term safe government payment promises. They also are of prime importance to the pricing of risky government payments. Intuitively, the further distant the payment is in time, the smaller the influence of current economic conditions on the future expected marginal utility of consumption. Hence, how one values future safe or risky (in our case, wage-based) assets depends primarily on one's current economic condition. If current times are good (current consumption is high), consumption in the future is more valuable because it has a higher marginal expect utility. Hence, a dollar of extra future consumption delivered via either a safe or risky asset is discounted (made less of) at a lower rate. If current times are bad, the opposite holds. Hence, discounting is highly sensitive to current conditions, not future economic risk.

We also find that agents value (discount) risky assets, whose risk is proportional to the wage, essentially identically to riskless assets. The reason is that wage-based securities are also, in our model, effectively zero-beta assets.

The model does, however, generate large risk premiums (i.e., much higher discount rates/lower prices) for government options, which promise payments only in good times. We also show that a) short-term bond market has an important, if small impact on pricing government obligations and b) at least in our context, APT pricing can approximate consumption-asset pricing reasonably well provided it is implemented using prevailing safe rates of return.

Finally, we demonstrate how one can derive discount rates appropriate to future generations, showing that their valuations of future government promises depends on the manner in which they are compensated for forgoing future government benefits.

Our paper's goal was modest – showing in a very simple framework that one can price securities in large-scale OLG models, which are buffeted by macro shocks, but whose financial markets are incomplete. Pricing safe and wage-based risk government obligations as well as government options are just two of a plethora of securities that can potentially be priced with the machinery presented here.

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